

A QUATERNIONIC BRAID REPRESENTATION (AFTER GOLDSCHMIDT AND JONES)

ERIC C. ROWELL

ABSTRACT. We show that the braid group representations associated with the $(3, 6)$ -quotients of the Hecke algebras factor over a finite group. This was known to experts going back to the 1980s, but a proof has never appeared in print. Our proof uses an unpublished quaternionic representation of the braid group due to Goldschmidt and Jones. Possible topological and categorical generalizations are discussed.

1. INTRODUCTION

Jones analyzed the images of the braid group representations obtained from Temperley-Lieb algebras in [11] where, in particular, he determined when the braid group images are finite or not. Braid group representations with finite image were also recognized in [12] and [8]. Some 15 years later the problem of determining the closure of the image of braid group representations associated with Hecke algebras played a critical role in analyzing the computational power of the topological model for quantum computation [6]. Following these developments the author and collaborators analyzed braid group representations associated with *BMW*-algebras [15] and twisted doubles of finite groups [5].

Partially motivated by empirical evidence the author conjectured that the braid group representations associated with an object X in a braided fusion category \mathcal{C} has finite image if, and only if, the Frobenius-Perron dimension of \mathcal{C} is integral (see eg. [25, Conjecture 6.6]). In [18, 24] various instances of this conjecture were verified. This current work verifies this conjecture for the braided fusion category $\mathcal{C}(\mathfrak{sl}_3, 6)$ obtained from the representation category of the quantum group $U_q \mathfrak{sl}_3$ at $q = e^{\pi i/6}$ (see [22] for details and notation).

More generally, Jimbo's [10] quantum Schur-Weyl duality establishes a relationship between the modular categories $\mathcal{C}(\mathfrak{sl}_k, \ell)$ obtained from the quantum group $U_q \mathfrak{sl}_k$ at $q = e^{\pi i/\ell}$ and certain semisimple quotients $\mathcal{H}_n(k, \ell)$ of specialized Hecke algebras $\mathcal{H}_n(q)$ (defined below). That is, if we denote by $X \in \mathcal{C}(\mathfrak{sl}_k, \ell)$ the simple

2000 *Mathematics Subject Classification.* Primary 20F36; Secondary 20C08, 57M25.

Key words and phrases. braid group, Hecke algebra, modular category.

The author was partially supported by NSA grant H98230-10-1-0215 and benefitted from discussions with V. F. R. Jones, M. J. Larsen, V. Ostrik, D. Nikshych, F. Sottile and H. Wenzl. Handwritten notes by D. Goldschmidt and V. Jones were very useful.

object analogous to the vector representation of \mathfrak{sl}_k then there is an isomorphism $\mathcal{H}_n(k, \ell) \cong \text{End}(X^{\otimes n})$ induced by $g_i \rightarrow I_X^{\otimes i-1} \otimes c_{X,X} \otimes I^{\otimes n-i-1}$.

In particular, the braid group representations associated with the modular category $\mathcal{C}(\mathfrak{sl}_3, 6)$ are the same as those obtained from $\mathcal{H}_n(3, 6)$. It is known that braid group representations obtained from $\mathcal{H}_n(3, 6)$ have finite image (mentioned in [6, 14, 18]) but a proof has never appeared in print. This fact was discovered by Goldschmidt and Jones during the writing of [8] and independently by Larsen during the writing of [6]. We benefitted from the notes of Goldschmidt and Jones containing the description of the quaternionic braid representation below. Our techniques follow closely those of [11, 12, 13].

The rest of the paper is organized into three sections. In Section 2 we recall some notation and facts about Hecke algebras and their quotients. The main results are in Section 3, and in Section 4 we indicate how the category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is exceptional from topological and categorical points of view.

2. HECKE ALGEBRAS

We extract the necessary definitions and results from [27] that we will need in the sequel.

Definition 2.1. The *Hecke algebra* $\mathcal{H}_n(q)$ for $q \in \mathbb{C}$ is the \mathbb{C} -algebra with generators g_1, \dots, g_{n-1} satisfying relations:

$$\begin{aligned} (H1)' \quad & g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq n-2 \\ (H2)' \quad & g_i g_j = g_j g_i \text{ for } |i-j| > 1 \\ (H3)' \quad & (g_i + 1)(g_i - q) = 0 \end{aligned}$$

Technically, $\mathcal{H}_n(q)$ is the Hecke algebra of type A , but we will not be considering other types so we suppress this distinction. One immediately observes that $\mathcal{H}_n(q)$ is the quotient of the braid group algebra $\mathbb{C}\mathcal{B}_n$ by the relation $(H3)'$. $\mathcal{H}_n(q)$ may also be described in terms of the generators $e_i := \frac{(q-g_i)}{(1+q)}$, which satisfy:

$$\begin{aligned} (H1) \quad & e_i^2 = e_i \\ (H2) \quad & e_i e_j = e_j e_i \text{ for } |i-j| > 1 \\ (H3) \quad & e_i e_{i+1} e_i - q/(1+q)^2 e_i = e_{i+1} e_i e_{i+1} - q/(1+q)^2 e_{i+1} \text{ for } 1 \leq i \leq n-2 \end{aligned}$$

For any $\eta \in \mathbb{C}$, Ocneanu [7] showed that one may uniquely define a linear functional tr on $\mathcal{H}_\infty(q) := \cup_{n=1}^\infty \mathcal{H}_n(q)$ satisfying

$$\begin{aligned} (1) \quad & \text{tr}(1) = 1 \\ (2) \quad & \text{tr}(ab) = \text{tr}(ba) \\ (3) \quad & \text{tr}(x e_n) = \eta \text{tr}(x) \text{ for any } x \in \mathcal{H}_n(q) \end{aligned}$$

Any linear function on \mathcal{H}_∞ satisfying these conditions is called a *Markov trace* and is determined by the value $\eta = \text{tr}(e_1)$.

Now suppose that $q = e^{2\pi i/\ell}$ and $\eta = \frac{(1-q^{1-k})}{(1+q)(1-q^k)}$ for some integers $k < \ell$. Then for each n , the (semisimple) quotient of $\mathcal{H}_n(q)$ by the annihilator of the restriction of

the trace $\mathcal{H}_n(q)/\text{Ann}(\text{tr})$ is called the (k, ℓ) -quotient. We will denote this quotient by $\mathcal{H}_n(k, \ell)$ for convenience. Wenzl [27] has shown that $\mathcal{H}_n(k, \ell)$ is semisimple and described the irreducible representations $\rho_\lambda^{(k, \ell)}$ where λ is a (k, ℓ) -admissible Young diagrams of size n . Here a Young diagram λ is (k, ℓ) -admissible if λ has at most k rows and $\lambda_1 - \lambda_k \leq \ell - k$ where λ_i denotes the number of boxes in the i th row of λ . The (faithful) Jones-Wenzl representation is the sum: $\rho^{(k, \ell)} = \bigoplus_\lambda \rho_\lambda^{(k, \ell)}$. Wenzl [27] has shown that $\rho^{(k, \ell)}$ is a C^* representation, i.e. the representation space is a Hilbert space (with respect to a Hermitian form induced by the trace tr) and $\rho_\lambda^{(k, \ell)}(e_i)$ is a self-adjoint operator. One important consequence is that each $\rho_\lambda^{(k, \ell)}$ induces an irreducible unitary representation of the braid group \mathcal{B}_n via composition with $\sigma_i \rightarrow g_i$, which is also called the Jones-Wenzl representation of \mathcal{B}_n .

3. A QUATERNIONIC REPRESENTATION

Consider the $(3, 6)$ -quotient $\mathcal{H}_n(3, 6)$. The $(3, 6)$ -admissible Young diagrams have at most 3 rows and $\lambda_1 - \lambda_3 \leq 3$. For $n \geq 3$ there are either 3 or 4 Young diagrams of size n that are $(3, 6)$ -admissible, and $\eta = \frac{(1-q^{1-3})}{(1+q)(1-q^3)} = 1/2$ in this case. Denote by ϕ_n the unitary Jones-Wenzl representation of \mathcal{B}_n induced by $\rho^{(3, 6)}$. Our main goal is to prove the following:

Theorem 3.1. *The image $\phi_n(\mathcal{B}_n)$ is a finite group.*

We will prove this theorem by embedding $\mathcal{H}_n(3, 6)$ into a finite dimensional algebra (Lemma 3.1) and then showing that the group generated by the images of g_1, \dots, g_{n-1} is finite (Lemma 3.2).

Denote by $[\cdot, \cdot]$ the multiplicative group commutator and let $q = e^{2\pi i/6}$. Consider the \mathbb{C} -algebra Q_n with generators $u_1, v_1, \dots, u_{n-1}, v_{n-1}$ subject to the relations:

- (G1) $u_i^2 = v_i^2 = -1$,
- (G2) $[u_i, v_j] = -1$ if $|i - j| \leq 1$,
- (G3) $[u_i, v_j] = 1$ if $|i - j| \geq 2$
- (G4) $[u_i, u_j] = [v_i, v_j] = 1$

Notice that the group $\{\pm 1, \pm u_i, \pm v_i, \pm u_i v_i\}$ is isomorphic to the group of quaternions. We see from these relations that $\dim(Q_n) = 2^{2n-2}$ since each word in the u_i, v_i has a unique normal form

$$(3.1) \quad \pm u_1^{\epsilon_1} \cdots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \cdots v_{n-1}^{\nu_{n-1}}$$

with $\nu_i, \epsilon_i \in \{0, 1\}$. Observe that a basis for Q_n is given by taking all $+$ signs in (3.1). We define a \mathbb{C} -valued trace Tr on Q_n by setting $\text{Tr}(1) = 1$ and $\text{Tr}(w) = 0$ for any non-identity word in the u_i, v_i . One deduces that Tr is faithful from the uniqueness of the normal form (3.1).

Define

$$(3.2) \quad s_i := \frac{-1}{2q}(1 + u_i + v_i + u_i v_i)$$

for $1 \leq i \leq n-1$.

Lemma 3.1. *The subalgebra $\mathcal{A}_n \subset Q_n$ generated by s_1, \dots, s_{n-1} is isomorphic to $\mathcal{H}_n(3, 6)$.*

Proof. It is a straightforward computation to see that the s_i satisfy

- (B1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
- (B2) $s_j s_i = s_i s_j$ if $|i - j| \geq 2$
- (E1) $(s_i - q)(s_i + 1) = 0$

Indeed, relation (B2) is immediate from relations (G3) and (G4). It is enough to check (B1) and (E1) for $i = 1$. For this we compute:

$$(3.3) \quad \begin{aligned} s_1^{-1} &= -\frac{q}{2}(1 - u_1 - v_1 - u_1 v_1) \\ s_1^{-1} u_1 s_1 &= u_1 v_1, \quad s_1^{-1} v_1 s_1 = u_1, \\ s_1^{-1} u_2 s_1 &= u_2 v_1, \quad s_1^{-1} v_2 s_1 = -u_1 v_1 v_2 \end{aligned}$$

from which (B1) and (E1) are deduced.

Thus $\varphi(g_i) = s_i$ induces an algebra homomorphism $\varphi : \mathcal{H}_n(q) \rightarrow Q_n$ with $\varphi(\mathcal{H}_n(q)) = \mathcal{A}_n$. Set $f_i := \varphi(e_i) = \frac{(q-s_i)}{(1+q)}$ and let $b \in Q_{n-1}$ that is, b is in the span of the words in $\{u_1, v_1, \dots, u_{n-2}, v_{n-2}\}$. The constant term of $f_{n-1}b$ is the product of the constant terms of b and f_{n-1} since f_{n-1} is in the span of $\{1, u_{n-1}, v_{n-1}, u_{n-1}v_{n-1}\}$, so $\text{Tr}(f_{n-1}b) = \text{Tr}(f_{n-1}) \text{Tr}(b)$. For each $a \in \mathcal{H}_n(q)$ we define $\varphi^{-1}(\text{Tr})(a) := \text{Tr}(\varphi(a))$, and conclude that $\varphi^{-1}(\text{Tr})$ is a Markov trace on $\mathcal{H}_n(q)$. Computing, we see that $\text{Tr}(f_{n-1}) = 1/2$ so that by uniqueness $\varphi^{-1}(\text{Tr}) = \text{tr}$ as functionals on $\mathcal{H}_n(q)$. Now if $a \in \ker(\varphi)$ we see that $\text{tr}(ac) = \text{Tr}(\varphi(ac)) = 0$ for any c so that $\ker(\varphi) \subset \text{Ann}(\text{tr})$. On the other hand, if $a \in \text{Ann}(\text{tr})$ we must have $\text{Tr}(\varphi(ac)) = \text{tr}(ac) = 0$ for all $c \in \mathcal{H}_n(q)$. If $\varphi(a) \neq 0$ then, by definition of Tr and φ , there exists an $a^\dagger \in \mathcal{H}_n(q)$ such that $\text{Tr}(\varphi(a)\varphi(a^\dagger)) \neq 0$ since Tr is faithful. Therefore $\text{Ann}(\text{tr}) \subset \ker(\varphi)$. In particular, we see that φ induces:

$$\mathcal{H}_n(3, 6) = \mathcal{H}_n(q) / \text{Ann}(\text{tr}) \cong \varphi(\mathcal{H}_n(q)) = \mathcal{A}_n \subset Q_n.$$

□

Lemma 3.2. *The group G_n generated by s_1, \dots, s_{n-1} is finite.*

Proof. Consider the conjugation action of the s_i on Q_n . We claim that the conjugation action of s_i on the words in the u_i, v_i is by a signed permutation. Since s_i commutes with words in u_j, v_j with $j \notin \{i-1, i, i+1\}$, by symmetry it is enough to consider the conjugation action of s_1 on the four elements $\{u_1, v_1, u_2, v_2\}$, which is given in (3.3).

Thus we see that G_n modulo the kernel of this action is a (finite) signed permutation group. The kernel of this conjugation action lies in the center $Z(Q_n)$ of Q_n . Using the normal form above we find that the center $Z(Q_n)$ is either 1-dimensional or 4-dimensional. Indeed, since the words:

$$W := \{u_1^{\epsilon_1} \cdots u_{n-1}^{\epsilon_{n-1}} v_1^{\nu_1} \cdots v_{n-1}^{\nu_{n-1}}\}$$

for $(\epsilon_1, \dots, \epsilon_{n-1}, \nu_1, \dots, \nu_{n-1}) \in \mathbb{Z}_2^{2n-2}$ form a basis for Q_n and $tw = \pm wt$ for $w, t \in W$ we may explicitly compute a basis for the center as those words $w \in W$ that commute with u_i and v_i for all i . This yields two systems of linear equations over \mathbb{Z}_2 :

$$(3.4) \quad \begin{cases} \epsilon_1 + \epsilon_2 = 0, \\ \epsilon_i + \epsilon_{i+1} + \epsilon_{i+2} = 0, & 1 \leq i \leq n-3 \\ \epsilon_{n-2} + \epsilon_{n-1} = 0 \end{cases}$$

and

$$(3.5) \quad \begin{cases} \nu_1 + \nu_2 = 0 \\ \nu_{i-1} + \nu_i + \nu_{i+1} = 0, & 1 \leq i \leq n-3 \\ \nu_{n-2} + \nu_{n-1} = 0. \end{cases}$$

Non-trivial solutions to (3.4) only exist if $3 \mid n$ since we must have $\epsilon_1 = \epsilon_2 = \epsilon_{n-2} = \epsilon_{n-1} = 1$ as well as $\epsilon_i = 0$ if $3 \mid i$ and $\epsilon_j = 1$ if $3 \nmid j$ and similarly for (3.5). Thus $Z(Q_n)$ is \mathbb{C} if $3 \nmid n$ and is spanned by $1, U, V$ and UV where $U = \prod_{3 \nmid i} u_i$ and $V = \prod_{3 \nmid i} v_i$ if $3 \mid n$. The determinant of the image of s_i under any representation is a 6th root of unity and hence the same is true for any element $z \in Z(Q_n) \cap G_n$. Thus for $3 \nmid n$ the image of any $z \in Z(Q_n) \cap G_n$ under the left regular representation is a root of unity times the identity matrix, and thus has finite order. Similarly, if $3 \mid n$, the restriction of any $z \in Z(Q_n) \cap G_n$ to any of the four simple components of the left regular representation is a root of unity times the identity matrix and so has finite order. So the group G_n itself is finite. \square

This completes the proof of Theorem 3.1.

Remark 3.1. The proof of Lemma 3.2 shows that the projective image of G_n is a (non-abelian) subgroup of the full monomial group $G(2, 1, 4^{n-1})$ of signed $4^{n-1} \times 4^{n-1}$ matrices. The main goal of this paper is to verify [25, Conjecture 6.6] in this case, but with further effort one could determine the group G_n more precisely. It is suggested in [14] that G_n is an extension of $PSU(n-1, \mathbb{F}_2)$ so that

$$|G_n| \approx \frac{1}{3} 2^{(n-1)(n-2)/2} \prod_{i=1}^{n-1} (2^i - (-1)^i)$$

but that such a result has not appeared in print. Modulo the center, the generators s_i have order 3 so that $G_n/Z(G_n)$ is a quotient of the factor group $\mathcal{B}_n/\langle \sigma_1^3 \rangle$ (here

σ_i are the usual generators of \mathcal{B}_n). For $n \leq 5$, Coxeter [1] has shown that these quotients are finite groups and determined their structure. In particular, the projective image of $\mathcal{B}_5/\langle\sigma_1^3\rangle$ is $PSU(4, \mathbb{F}_2)$, so G_5 is an extension of this simple group. A strategy for showing G_n is an extension of $PSU(n-1, \mathbb{F}_2)$ for $n > 5$ would be to find an $(n-1)$ -dimensional invariant subspace of Q_n so that the restricted action of the braid generators is by order 3 pseudo-reflections (projectively). A comparison of the dimensions of the simple $\mathcal{H}_n(2, 6)$ -modules with those of $PSU(n-1, \mathbb{F}_2)$ indicates that one must also restrict to those n not divisible by 3.

4. CONCLUDING REMARKS, QUESTIONS AND SPECULATIONS

The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ does not seem to have any obvious generalizations. We discuss some of the ways in which $\mathcal{C}(\mathfrak{sl}_3, 6)$ appears to be exceptional by posing a number of (somewhat naïve) questions which we expect to have negative answers.

4.1. Link Invariants. From any modular category one obtains (quantum) link invariants via Turaev's approach [26]. The link invariant $P'_L(q, \eta)$ associated with $\mathcal{C}(\mathfrak{sl}_k, \ell)$ is (a variant of) the HOMFLY-PT polynomial ([7], where a different choice of variables is used). For the choices $q = e^{2\pi i/6}$ and $\eta = 1/2$ corresponding to $\mathcal{C}(\mathfrak{sl}_3, 6)$ the invariant has been identified [16]:

$$P'_L(e^{2\pi i/6}, 1/2) = \pm i(\sqrt{2})^{\dim H_1(T_L; \mathbb{Z}_2)}$$

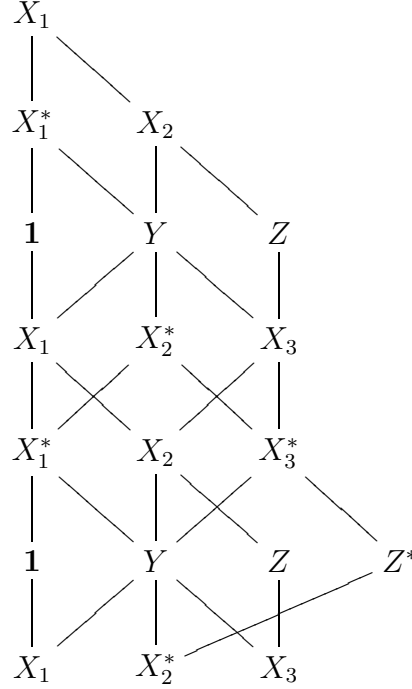
where T_L is the triple cyclic cover of the three sphere S^3 branched over the link L . There is a similar series of invariants for any odd prime p : $\pm i(\sqrt{p})^{\dim H_1(D_L; \mathbb{Z}_p)}$ where D_L is the double cyclic cover of S^3 branched over L (see [16, 8]). It appears that this series of invariants can be obtained from modular categories $\mathcal{C}(\mathfrak{so}_p, 2p)$. This has been verified for $p = 3, 5$ (see [8] and [12]) and we have recently handled the $p = 7$ case (unpublished, using results in [29]).

Question 4.1. *Are there modular categories with associated link invariant:*

$$\pm i(\sqrt{p})^{\dim H_1(T_L; \mathbb{Z}_p)}?$$

In [15] it is suggested that if the braid group images corresponding to some ribbon category are finite then the corresponding link invariant is *classical*, i.e. equivalent to a homotopy-type invariant. Another formulation of this idea is found in [23] in which *classical* is interpreted in terms of computational complexity.

4.2. Fusion Categories and II_1 Factors. The category $\mathcal{C}(\mathfrak{sl}_3, 6)$ is an *integral* fusion category, that is, the simple objects have integral dimensions. The categories $\mathcal{C}(\mathfrak{sl}_k, \ell)$ are integral for $(k, \ell) = (3, 6)$ and $(k, k+1)$ but no other examples are known (or believed to exist). $\mathcal{C}(\mathfrak{sl}_3, 6)$ has six simple (isomorphism classes of) objects: $\{X_i, X_i^*\}_{i=1}^3$ of dimension 2 (dual pairs), three simple objects $\mathbf{1}, Z, Z^*$ of dimension 1, and one simple object Y of dimension 3. The Bratteli diagram for tensor powers of the generating object X_1 is given in Figure 1. It is shown in [4] that

FIGURE 1. Bratteli diagram for $\mathcal{C}(\mathfrak{sl}_3, 6)$

\mathcal{C} is an integral fusion category if, and only if, $\mathcal{C} \cong \text{Rep}(H)$ for some semisimple finite dimensional quasi-Hopf algebra H , so in particular $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \text{Rep}(H)$ for some quasi-triangular quasi-Hopf algebra H . One wonders if strict coassociativity can be achieved:

Question 4.2. *Is there a (quasi-triangular) semisimple finite dimensional Hopf algebra H with $\mathcal{C}(\mathfrak{sl}_3, 6) \cong \text{Rep}(H)$?*

Other examples of integral categories are the representation categories $\text{Rep}(D^\omega G)$ of twisted doubles of finite groups studied in [5] (here G is a finite group and ω is a 3-cocycle on G). Any fusion category \mathcal{C} with the property that its Drinfeld center $\mathcal{Z}(\mathcal{C})$ is equivalent as a braided fusion category to $\text{Rep}(D^\omega G)$ for some ω, G is called *group-theoretical* (see [4, 19]). The main result of [5] implies that if \mathcal{C} is any braided group-theoretical fusion category then the braid group representations obtained from \mathcal{C} must have finite image. In [18] we showed that $\mathcal{C}(\mathfrak{sl}_3, 6)$ is not group-theoretical and in fact has minimal dimension (36) among non-group-theoretical integral modular categories.

Question 4.3. *Is there a family of non-group-theoretical integral modular categories that includes $\mathcal{C}(\mathfrak{sl}_3, 6)$?*

Notice that $\mathcal{C}(\mathfrak{sl}_3, 6)$ has a ribbon subcategory \mathcal{D} with simple objects $\mathbf{1}, Z, Z^*$ and Y . The fusion rules are the same as those of $\text{Rep}(\mathfrak{sl}_4)$: $Y \otimes Y \cong \mathbf{1} \oplus Z \oplus$

$Z^* \oplus Y$. However \mathcal{D} is not symmetric, and $\mathcal{C}(\mathfrak{sl}_3, 6)$ has smallest dimension among modular categories containing \mathcal{D} as a ribbon subcategory (what Müger would call a *minimal modular extension* [17]). One possible generalization of $\mathcal{C}(\mathfrak{sl}_3, 6)$ would be a minimal modular extension of a non-symmetric ribbon category \mathcal{D}_n similar to \mathcal{D} above. That is, \mathcal{D}_n should be a non-symmetric ribbon category with n 1-dimensional simple objects $\mathbf{1} = Z_0, \dots, Z_{n-1}$ and one simple n -dimensional object Y_n such that $Y_n \otimes Y_n \cong Y_n \oplus \bigoplus_{i=0}^{n-1} Z_i$ and the Z_i have fusion rules like \mathbb{Z}_n . For \mathcal{D}_n to exist even at the generality of fusion categories one must have $n = p^\alpha - 1$ for some prime p and integer α by [3, Corollary 7.4]. However, V. Ostrik [20] informs us that these categories do not admit non-symmetric braidings except for $n = 2, 3$. So this does not produce a generalization.

A pair of hyperfinite II_1 factors $A \subset B$ with index $[B : A] = 4$ can be constructed from $\mathcal{C}(\mathfrak{sl}_3, 6)$ (see [28, Section 4.5]). The corresponding principal graph is the Dynkin diagram $E_6^{(1)}$ the nodes of which we label by simple objects:

$$\begin{array}{ccccccc}
 & & & & Z^* & & \\
 & & & & | & & \\
 & & & & X_3 & & \\
 & & & & | & & \\
 \mathbf{1} & \text{---} & X_1 & \text{---} & Y & \text{---} & X_2^* \text{---} Z
 \end{array}$$

This principal graph can be obtained directly from Bratteli diagram in Figure 1 as the nodes in the 6th and 7th levels and the edges between them. Hong [9] showed that any II_1 subfactor pair $M \subset N$ with principal graph $E_6^{(1)}$ can be constructed from some II_1 factor P with an outer action of \mathfrak{A}_4 as $M = P \rtimes \mathbb{Z}_3 \subset P \rtimes \mathfrak{A}_4 = N$. Subfactor pairs with principal graph $E_7^{(1)}$ and $E_8^{(1)}$ can also be constructed (see eg. [21]). We ask:

Question 4.4. *Is there a unitary non-group-theoretical integral modular category with principal graph $E_7^{(1)}$ or $E_8^{(1)}$?*

Even a braided fusion category with such a principal graph would be interesting, and have interesting braid group image.

Notice that the subcategory \mathcal{D} mentioned above plays a role here as \mathfrak{A}_4 corresponds to the Dynkin diagram $E_6^{(1)}$ in the McKay correspondence. A modular category \mathcal{C} with principal graph $E_7^{(1)}$ (resp. $E_8^{(1)}$) would contain a ribbon subcategory \mathcal{F}_1 (resp. \mathcal{F}_2) with the same fusion rules as $\text{Rep}(\mathfrak{S}_4)$ (resp. $\text{Rep}(\mathfrak{A}_5)$). Using [2, Lemma 1.2] we find that such a category \mathcal{C} must have dimension divisible by 144 (resp. 3600). The ribbon subcategory \mathcal{F}_2 must have symmetric braiding (D. Nikshych's proof: $\text{Rep}(\mathfrak{A}_5)$ has no non-trivial fusion subcategories so if it has a non-symmetric braiding, the Müger center is trivial. But if the Müger center is

trivial it is modular, which fails by [2, Lemma 1.2]). This suggests that for $E_8^{(1)}$ the answer to Question 4.4 is “no.” There is a non-symmetric choice for \mathcal{F}_1 (as V. Ostrik informs us [20]), with Müger center equivalent to $\text{Rep}(\mathfrak{S}_3)$. By [17, Prop. 5.1] a minimal modular extension \mathcal{C} of such an \mathcal{F}_1 would have dimension 144.

REFERENCES

- [1] H. S. M. Coxeter, *Factor Groups of the Braid Group*. Proceedings of the Fourth Can. Math. Congress, Banff 1957, University of Toronto Press, (1959), 95–122.
- [2] P. Etingof and S. Gelaki, *Some properties of finite-dimensional semisimple Hopf algebras*, Mathematical Research Letters **5** (1998), 191–197.
- [3] P. Etingof, S. Gelaki and V. Ostrik, *Classification of fusion categories of dimension pq* . Int. Math. Res. Not. **2004**, no. 57, 3041–3056.
- [4] P. Etingof, D. Nikshych, and V. Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), no. 2, 581–642.
- [5] P. Etingof, E. C. Rowell and S. J. Witherspoon, *Braid group representations from quantum doubles of finite groups*, Pacific J. Math. **234**, no. 1 (2008), 33–41.
- [6] M. H. Freedman, M. J. Larsen and Z. Wang, *The two-eigenvalue problem and density of Jones representation of braid groups*, Comm. Math. Phys. **228** (2002), 177–199.
- [7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett and A. Ocneanu, *A new polynomial invariant of knots and links*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), no. 2, 239–246.
- [8] D. M. Goldschmidt and V. F. R. Jones, *Metaplectic link invariants*, Geom. Dedicata **31** (1989), no. 2, 165–191.
- [9] J. H. Hong, *Subfactors with principal graph $E_6^{(1)}$* , Acta Appl. Math. **40** (1995), no. 3, 255–264.
- [10] M. Jimbo, *A q -analogue of $U(\mathfrak{gl}(N+1))$, Hecke algebra, and the Yang-Baxter equation*, Lett. Math. Phys. **11** (1986), no. 3, 247–252.
- [11] V. F. R. Jones, *Braid groups, Hecke algebras and type II_1 factors*, *Geometric methods in operator algebras (Kyoto, 1983)*, 242–273, Pitman Res. Notes Math. Ser., 123, Longman Sci. Tech., Harlow, 1986.
- [12] V. F. R. Jones, *On a certain value of the Kauffman polynomial*, Comm. Math. Phys. **125** (1989), no. 3, 459–467.
- [13] M. J. Larsen, E. C. Rowell, *An algebra-level version of a link-polynomial identity of Lickorish*. Math. Proc. Cambridge Philos. Soc. **144** no. 3 (2008) 623–638.
- [14] M. J. Larsen and E. C. Rowell, *Unitary braid group representations with finite image*, Algebr. Geom. Topol. **8** (2008), no. 4, 2063–2079.
- [15] M. J. Larsen, E. C. Rowell and Z. Wang, *The N -eigenvalue problem and two applications*, Int. Math. Res. Not. **2005** (2005), no. 64, 3987–4018.
- [16] W. B. R. Lickorish and K. C. Millett, *Some evaluations of link polynomials*, Comment. Math. Helvetici **61** (1986) 349–359.
- [17] M. Müger, *On the structure of modular categories*, Proc. London Math. Soc. (3) **87** (2003), 291–308.
- [18] D. Naidu and E. C. Rowell, *A finiteness property for braided fusion categories*, to appear in Algebr. Represent. Theory.
- [19] S. Natale *On group theoretical Hopf algebras and exact factorizations of finite groups*, J. Algebra **270** (2003), no. 1, 199–211.
- [20] V. Ostrik, private communication.

- [21] S. Popa, *Sur la classification des sousfacteurs d'indice fini des facteur hyperfini*, C. R. Acad. Sci. Paris **311** (1990), 95-100.
- [22] E. C. Rowell, *From quantum groups to unitary modular tensor categories*, in Contemp. Math. **413** (2006), 215-230.
- [23] E. C. Rowell, *Two paradigms for topological quantum computation*, in Contemp. Math. **482** (2009), 165-177.
- [24] E. C. Rowell, *Braid representations from quantum groups of exceptional Lie type*, Rev. Un. Mat. Argentina **51** (2010) no. 1, 165-175.
- [25] E. Rowell, R. Stong and Z. Wang, *On classification of modular tensor categories*, Comm. Math. Phys. **292** (2009) no. 2, 343-389.
- [26] V. G. Turaev, *Quantum Invariants of Knots and 3-Manifolds*. de Gruyter Studies in Mathematics, 18. Walter de Gruyter & Co., Berlin, 1994.
- [27] H. Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math. **92** (1988), no. 2, 349-383.
- [28] H. Wenzl, *C^* tensor categories from quantum groups*, J. Amer. Math. Soc. **11** (1998), no. 2, 261-282.
- [29] B. W. Westbury, *Invariant tensors for the spin representation of $\mathfrak{so}(7)$* , Math. Proc. Cambridge Philos. Soc. **144** (2008), no. 1, 217-240.

DEPARTMENT OF MATHEMATICS, TEXAS A&M UNIVERSITY, COLLEGE STATION, TX 77843,
U.S.A.

E-mail address: rowell@math.tamu.edu